$$\frac{d^2 V(x,t)}{dx^2} = LC\frac{d^2 V(x,t)}{dt^2} + (RC+GL)\frac{dV(x,t)}{dt} + RG\,V(x,t) \quad -(5)$$

Similarly for i

$$\frac{d^2 I(x,t)}{dx^2} = LC\frac{d^2 I(x,t)}{dt^2} + (RC+GL)\frac{dI(x,t)}{dt} + RG\,i(x,t) \quad -(6)$$

eqn. (5) & (6) can be generalized -

$$\left.\begin{aligned} \frac{d^2 V(x)}{dx^2} - \gamma^2 V(x) &= 0 \\ \frac{d^2 I(x)}{dx^2} - \gamma^2 I(x) &= 0 \end{aligned}\right\} -(A)$$

where $\gamma = \alpha + j\beta$

$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$

γ = Propagation Constant

α = Attenuation constant

β = Phase Constant.

The soln. of differential eqn (A) is of exponential type

$$V(x) = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$$

$$I(x) = \frac{V^+ e^{-\gamma x} - V^- e^{+\gamma x}}{Z_0}$$

where Z_0 = characteristic imp.

$$= \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

Consider function

$f(x) \Rightarrow f(x-x_0)$ is shifted to right a distance x_0 along $+x$ direct.

$f(x-ut)$ is fun. $f(x)$ shifted to right of distance $x_0 = ut$ u = velocity, t = time.

Assume $f(x)$ a Sinusoidal function

$f(x) = A\cos\beta x$ A = Amplitude. β = Phase const.

Time-domain form.

$$f(x,t) = A\cos\beta(x-ut)$$

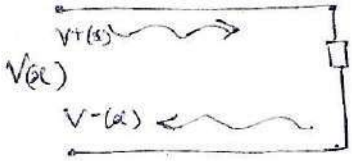
$$= A\cos(\beta x - \omega t) \qquad \omega = \beta \cdot u.$$

Ph. form. $F = A e^{-j\beta x}$.

We have $V(x) = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$

$V^+(x) = V^+ e^{-\gamma x}$

& $V^-(x) = V^- e^{+\gamma x}$.



Reflection Coefficient $\Gamma(x)$ is defined as ratio of reflected wave phasor to incident wave phasor.

$$\Gamma(x) = \frac{V^-(x)}{V^+(x)} = \frac{V^- e^{+\gamma x}}{V^+ e^{-\gamma x}} = \frac{V^-}{V^+} e^{2\gamma x} = \Gamma_L e^{2\gamma x}$$

Where $\cancel{\Gamma(L)}$ $\Gamma_L = \Gamma(0) = \frac{V^-}{V^+}$ is reflection coefficient at load end & x in the

$$Z_{IN}(x) = \frac{V(x)}{I(x)} = Z_0 \frac{e^{-\gamma x} + \Gamma_L e^{\gamma x}}{e^{-\gamma x} - \Gamma_L e^{\gamma x}}.$$

$\Gamma_L = \frac{V^-}{V^+}$

$V^- = \Gamma_L V^+$

$$V(x) = V^+ e^{-\gamma x} + \Gamma_L V^+ e^{+\gamma x}$$

$$= \frac{\frac{V^+(e^{-\gamma x} + \Gamma_L e^{+\gamma x})}{1}}{\frac{V^+(e^{-\gamma x} - \Gamma_L e^{+\gamma x})}{Z_0}}$$

$$= \frac{V^+(e^{-\gamma x} + \Gamma_L e^{+\gamma x})}{1} \frac{Z_0}{V^+(e^{-\gamma x} - \Gamma_L e^{+\gamma x})}$$

$$= Z_0 \frac{e^{-\gamma x} + \Gamma_L e^{+\gamma x}}{e^{-\gamma x} - \Gamma_L e^{+\gamma x}}$$

At load end where $x = 0$.

$$Z_{IN}(0) = Z_L = Z_0 \cdot \frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$$VSWR = \frac{V_{max}}{V_{min}}$$

$$V_{max} = |V(x)|_{max} = |V_0^+| + |V_0^-|$$
$$= |V_0^+|(1+|\Gamma_L|)$$

$$V_{min} = |V(x)|_{min} = |V_0^+| - |V_0^-|$$
$$= |V_0^+|(1-|\Gamma_L|)$$

$$\boxed{\therefore VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}}$$

The solⁿ. to second-order differential eqⁿ.

$$V(x) = V^+ e^{-\gamma x} + V^- e^{+\gamma x}$$

Where complex constants V^+ & V^- are determined by boundary conditions.

$$I(x) = \frac{V^+ e^{-\gamma x} - V^- e^{+\gamma x}}{Z_0}$$

Where $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$ is called characteristic impedance of transmission line.

For lossless transmission line

we have $R = G = 0$.

$$\therefore \gamma = j\omega\sqrt{LC} = j\beta$$

$$Z_0 = \sqrt{L/C}$$

where $\beta = \omega\sqrt{LC}$ is phase constant.

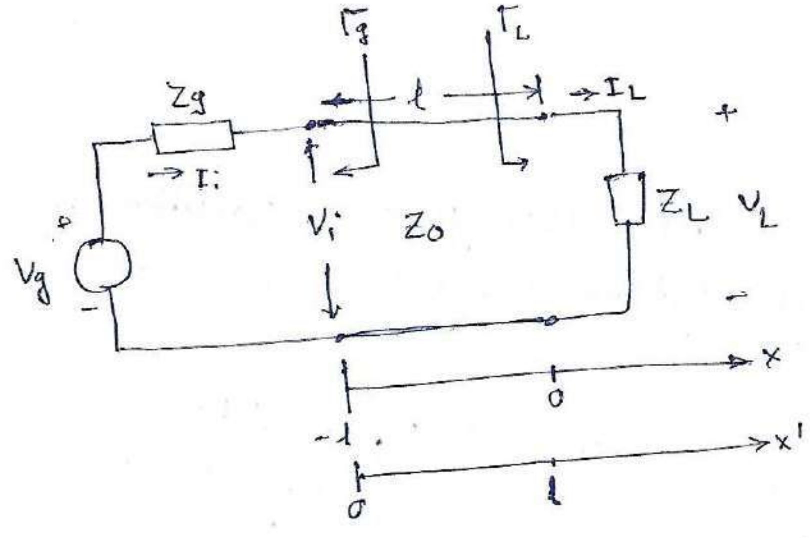
$$\therefore \frac{d^2V(x)}{dx^2} + \beta^2 V(x) = 0$$

$$\frac{d^2I(x)}{dx^2} + \beta^2 I(x) = 0$$

$$\Gamma_L = \frac{V^-}{V^+}$$

$$\therefore V(x) = V^+ e^{-j\beta x} + V^- e^{j\beta x} \Rightarrow V(x) = V^+ e^{-j\beta x}[1 + \Gamma(x)]$$

$$I(x) = \frac{V^+ e^{-j\beta x} - V^- e^{j\beta x}}{Z_0}.$$



The boundary conditions at each end can be written as.

1 B.C. #1. Voltage & current at $x=-l$ is given by

$$V_i = V_g - Z_g I_i$$

(2) B.C. #2 Voltage & Current at $x=0$ is given by

$$V_L = I_L \cdot Z_L$$

3. B.C. #3 Voltage & current on T.L. for $-l \leq x \leq 0$, is given by.

$$V(x) = V^+(x) + V^-(x)$$

$$= V_0^+ e^{-j\beta x}[1+\Gamma(x)] \quad \text{—— (1)}$$

$$I(x) = I_0^+(x) - I_0^-(x) = \frac{V_0^+}{Z_0} e^{-j\beta x}[1-\Gamma(x)] \quad \text{— (2)}$$

Where $\Gamma(x) = \dfrac{V^-(x)}{V^+(x)} = \dfrac{V_0^- e^{j\beta x}}{V_0^+ e^{-j\beta x}} = \dfrac{V_0^-}{V_0^+} e^{2j\beta x} = \Gamma_L e^{2j\beta x}$

Applying BC #1 & solve for $|V^+|$

$$V_i = V(-l) = V_0^+ e^{j\beta l}[1+\Gamma(-l)] = V_g - Z_g \cdot \frac{V_0^+}{Z_0} e^{j\beta l}[1-\Gamma(-l)] \quad \text{— (3)}$$

$$\text{Where } \Gamma(-l) = \Gamma_L e^{-j2\beta l}$$

Solve eq. (3) for V_0^+ in terms of V_g.

$$V_0^+ = \frac{Z_0 V_g}{Z_0 + Z_g}\left(\frac{e^{-j\beta l}}{1-\Gamma_L \Gamma_g e^{-2j\beta l}}\right) \quad \text{—— (4)}$$

Where $\Gamma_g = \dfrac{Z_g - Z_0}{Z_g + Z_0}$

Thus $V(x)$ & $I(x)$ under this general condition may be obtained by substituting for V_0^+ from eq. (3) into eq. (1) & (2)

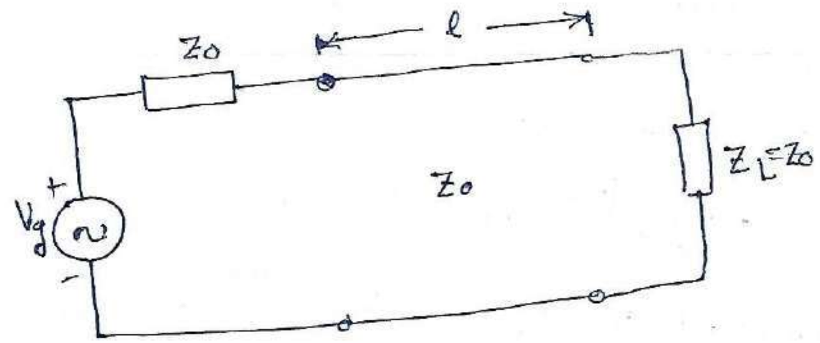
$$V(x) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta(x+l)} \left(\frac{1 + \Gamma_L e^{j2\beta x}}{1 - \Gamma_L \Gamma_g e^{-j2\beta l}} \right) \quad \text{— (A)}$$

$$I(x) = \frac{V_g}{Z_0 + Z_g} e^{-j\beta(x+l)} \left(\frac{1 - \Gamma_L e^{j2\beta x}}{1 - \Gamma_L \Gamma_g e^{-j2\beta l}} \right) \quad \text{— (B)}$$

Case-I - T.L. Matched at both ends.

$Z_g = Z_L = Z_0$

$\therefore \Gamma_g = \Gamma_L = 0$



$$\therefore V(x) = \frac{V_g}{2} e^{-j\beta(x+l)}$$

$$I(x) = \frac{V_g}{2Z_0} e^{-j\beta(x+l)}$$

At the generator end $x = -l$

$$V(-l) = V_i = \frac{V_g}{2} \quad \& \quad I(-l) = I_i = \frac{V_g}{2Z_0}$$

At the load end $x = 0$

$$V(0) = V_L = \frac{V_g}{2} e^{-j\beta l}$$

$$I(0) = I_L = \frac{V_g}{2Z_0} e^{-j\beta l}.$$

Case-II - Matched at the source end only

$$Z_g = Z_0, \quad Z_L \neq Z_0$$

$$\Gamma_g = 0 \quad \Gamma_L \neq 0$$

$$V(x) = \frac{V_g}{2} e^{-j\beta(x+l)} \left(1 + \Gamma_L e^{j2\beta x}\right)$$

$$I(x) = \frac{V_g}{2Z_0} e^{-j\beta(x+l)} \left(1 - \Gamma_L e^{j2\beta x}\right)$$

At generator end $x = -l$

$$V(-l) = V_i = \frac{V_g}{2}\left(1 + \Gamma_L e^{-2j\beta l}\right)$$

$$I(-l) = I_i = \frac{V_g}{2Z_0}\left(1 - \Gamma_L e^{-j2\beta l}\right)$$

At load end $x = 0$

$$V(0) = V_L = \frac{V_g}{2} e^{-j\beta l}(1 + \Gamma_L) \quad \& \quad I(0) = I_L = \frac{V_g}{2Z_0} e^{-j\beta l}(1 - \Gamma_L)$$

Case-III - Matched at the load end only.

$$Z_g \neq Z_0, \quad Z_L = Z_0, \quad \therefore \Gamma_g \neq 0, \quad \Gamma_L = 0$$

$$V(x) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta(x+l)} \quad \& \quad I(x) = \frac{V_g}{Z_0 + Z_g} e^{-j\beta(x+l)}$$

At generator end $x = -l$

$$V(-l) = V_i = \frac{Z_0 V_g}{Z_0 + Z_g}$$

$$I(-l) = I_i = \frac{V_g}{Z_0 + Z_g}$$

At load end $x = 0$

$$V(0) = V_L = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta l}$$

$$I(0) = I_L = \frac{V_g}{Z_0 + Z_g} e^{-j\beta l}$$